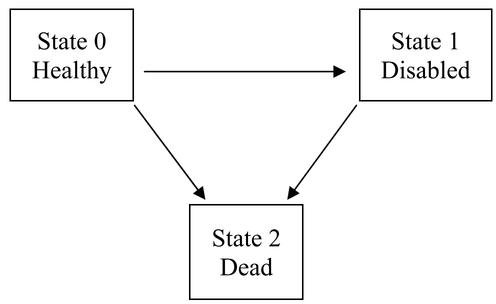
November 2013 Course MLC Examination, Problem No. 10

For a multiple state model, you are given:

(i)



(ii) The following forces of transition: $\mu^{01} = 0.02$, $\mu^{02} = 0.03$, $\mu^{12} = 0.05$.

(iii) Calculate the conditional probability that a Healthy life on January 1, 2004 is still Healthy on January 1, 2014, given that this person is not Dead on January 1, 2014.

A. 0.61 B. 0.68 C. 0.74 D. 0.79 E. 0.83

Solution.

Let H be the event that the life considered in this problem is Healthy on January 1, 2014, let E be the event that the same life is Disabled, and D be the event that the life is dead. The probability we are looking for is

$$\Pr\left(H \middle| D^C\right) = \Pr\left(H \middle| H \cup E\right) = \frac{\Pr\left(H \cap \left(H \cup E\right)\right)}{\Pr\left(H \cup E\right)} = \frac{\Pr\left(H\right)}{\Pr\left(H\right) + \Pr\left(E\right)}.$$

We also have $\Pr\left(H\right) = {}_{10}p^{00}$ and $\Pr\left(E\right) = {}_{10}p^{01}$. The transition matrix for this model is

$${}_rP = \begin{bmatrix} {}_rp^{00} & {}_rp^{01} & {}_rp^{02} \\ {}_rp^{10} & {}_rp^{11} & {}_rp^{12} \\ {}_rp^{20} & {}_rp^{21} & {}_rp^{22} \end{bmatrix}$$

and the generator matrix is

$$Q_t = \begin{bmatrix} -\left(\mu^{01} + \mu^{02}\right) & \mu^{01} & \mu^{02} \\ \mu^{10} & -\left(\mu^{10} + \mu^{12}\right) & \mu^{12} \\ \mu^{20} & \mu^{21} & -\left(\mu^{20} + \mu^{21}\right) \end{bmatrix} = \begin{bmatrix} -0.05 & 0.02 & 0.03 \\ 0 & -0.05 & 0.05 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the Kolmogorov Forward Equation is

$$\frac{d}{dr}\begin{bmatrix} {}_rp_t^{00} & {}_rp_t^{01} & {}_rp_t^{00} \\ {}_rp_t^{10} & {}_rp_t^{11} & {}_rp_t^{12} \\ {}_rp_t^{20} & {}_rp_t^{21} & {}_rp_t^{22} \end{bmatrix} = \begin{bmatrix} {}_rp_t^{00} & {}_rp_t^{01} & {}_rp_t^{00} \\ {}_rp_t^{10} & {}_rp_t^{11} & {}_rp_t^{12} \\ {}_rp_t^{20} & {}_rp_t^{21} & {}_rp_t^{22} \end{bmatrix} \cdot \begin{bmatrix} -0.05 & 0.02 & 0.03 \\ 0 & -0.05 & 0.05 \\ 0 & 0 & 0 \end{bmatrix}$$

The resulting individual equations are:

$$\frac{d}{dr}{}_rp^{00} = -0.05{}_rp^{00}, \quad \frac{d}{dr}{}_rp^{01} = 0.02{}_rp^{00} - 0.05p^{01}, \quad \frac{d}{dr}{}_rp^{02} = 0.03{}_rp^{00} + 0.05{}_rp^{01},$$
$$\frac{d}{dr}{}_rp^{10} = -0.05{}_rp^{10}, \quad \frac{d}{dr}{}_rp^{11} = 0.02{}_rp^{10} - 0.05{}_rp^{11}, \quad \frac{d}{dr}{}_rp^{12} = 0.03{}_rp^{10} + 0.05{}_rp^{11},$$
$$\frac{d}{dr}{}_rp^{20} = -0.05{}_rp^{20}, \quad \frac{d}{dr}{}_rp^{21} = 0.02{}_rp^{20} - 0.05{}_rp^{21}, \quad \frac{d}{dr}{}_rp^{22} = 0.03{}_rp^{20} + 0.05{}_rp^{21}.$$

The particular equation that relates to the quantity sought is

$$\frac{d}{dr}{}_rp^{01} = 0.02{}_rp^{00} - 0.05p^{01},$$

and this equation involves the probability sought ${}_rp^{01}$ and another probability ${}_rp^{00}$, for which we know that $\frac{d}{dr}{}_rp^{00} = -0.05{}_rp^{00}$, resulting in

$$\frac{\frac{d}{dr}{}_rp^{00}}{{}_rp^{00}} = \frac{d}{dr}\left(\ln {}_rp^{00}\right) = -0.05,$$

so that

$$\ln {}_rp^{00} = -0.05r + C,$$

or ${}_rp^{00} = e^C \cdot e^{-0.05r}$. But when $r = 0$, ${}_rp_t^{00} = {}_0p_t^{00} = 1$, so that $C = 0$. We conclude that ${}_rp^{00} = e^{-0.05r}$. In particular $\Pr(H) = {}_{10}p^{00} = e^{-0.5}$. Thus

$$\frac{d}{dr}{}_rp^{01} = 0.02e^{-0.05r} - 0.05{}_rp^{01},$$

or

$$\frac{d}{dr}{}_rp^{01} + 0.05{}_rp^{01} = 0.02e^{-0.05r}.$$

This is a first order non-homogeneous differential equation, where you pursue a solution with the use of an integrating factor $e^{\int_0^r 0.05\,dt} = e^{0.05r}$, which we multiply both sides of it by and obtain

$$e^{0.05r} \cdot \frac{d}{dr}{}_rp^{01} + \underbrace{0.05 \cdot e^{0.05r}}_{=\frac{d}{dr}e^{0.05r}} \cdot {}_rp^{01} = 0.02,$$

or $\frac{d}{dr}\left(e^{0.05r} \cdot {}_rp^{01}\right) = 0.02$, resulting in $e^{0.05r} \cdot {}_rp^{01} = 0.02r + C$. But we know that when r = 0, ${}_0p^{01} = 0$, so that $C = 0$. This means that ${}_rp^{01} = 0.02re^{-0.05r}$. In particular,

$$\Pr(E) = {}_{10}p^{01} = 0.2e^{-0.5}.$$

We conclude that

$$\Pr\left(H \middle| D^C\right) = \frac{\Pr(H)}{\Pr(H) + \Pr(E)} = \frac{{}_{10}p^{00}}{{}_{10}p^{00} + {}_{10}p^{01}} = \frac{e^{-0.5}}{e^{-0.5} + 0.2e^{-0.5}} = \frac{1}{1.2} = \frac{5}{6} \approx 0.8333.$$

Answer E.